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ON GAUSS'S METHOD OF SUBSTITUTION.

By Prof. W. Woolsey Johnson, Annapolis, Md.

In Gauss's notation the normal equations arising in the method of Least Squares, for μ unknown quantities $x, y, z, \ldots, t$, are

$$\left.\begin{array}{l} [aa]x + [ab]y + [ac]z + \ldots + [al]t = [an] \\ [ab]x + [bb]y + [bc]z + \ldots + [bl]t = [bn] \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ [al]x + [bl]y + [cl]z + \ldots + [ll]t = [ln] \end{array}\right\}, \tag{1}$$

and the result of eliminating x by means of the first normal equation are written

$$\left.\begin{array}{l} [bb, 1]y + [bc, 1]z + \ldots + [bl, 1]t = [bn, 1] \\ [bc, 1]y + [cc, 1]z + \ldots + [cl, 1]t = [cn, 1] \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ [bl, 1]y + [cl, 1]z + \ldots + [ll, 1]t = [ln, 1] \end{array}\right\}. \tag{2}$$

The fact that these "reduced normal equations," like the normal equations (1), have a symmetrical determinant suggests the question, May they not be regarded as normal equations derivable from certain fictitious observation equations for the $\mu - 1$ unknown quantities $y, z, \ldots, t$?

Oppolzer in fact raises the question directly ("Lehrbuch zur Bahnbestimmung der Kometen und Planeten," vol. ii, p. 331) "whether the newly introduced symbols $[bb, 1]$, $[bc, 1]$, ... may be regarded, in analogy with the symbols $[aa]$, $[ab]$, ..., as sums of products of quantities related to each other in the same way;" and he answers this question by finding $\frac{1}{2}m(m-1)$ quantities (where m is the number of observation equations) corresponding to each letter $b, c, \ldots, l$, of which $[bb, 1]$, $[bc, 1]$, ... are the sums of products, (and of squares when the letters of the symbol are alike). From this he proceeds to draw "the important inference that $[bb, 1]$, $[cc, 1]$, etc. are positive.

The fictitious observation equations which most naturally suggest themselves would seem to be the m equations which result from the elimination of x from the observation equations by means of the normal equation for x. That these equations, in fact, fulfil the requirements is readily seen; for,

writing them in the form

$$\left.\begin{array}{l} b'_1 y + c'_1 z + \ldots + l'_1 t = n'_1 \\ b'_2 y + c'_2 z + \ldots + l'_2 t = n'_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b'_\mu y + c'_\mu z + \ldots + l'_\mu t = n'_\mu \end{array}\right\}, \tag{3}$$

their coefficients and absolute terms are of the form

$$b' = b - \frac{[ab]}{[aa]}a\,, \qquad c' = c - \frac{[ac]}{[aa]}a\,, \qquad \ldots;$$

whence, for example,

$$[b'c'] = \Sigma bc - \frac{[ab]}{[aa]}\Sigma ac - \frac{[ac]}{[aa]}\Sigma ab + \frac{[ab]\,[ac]}{[aa]^2}\Sigma a^2$$

$$= [bc] - \frac{[ab]\,[ac]}{[aa]} = [bc, 1]. \tag{4}$$

So far as I know, this method of viewing the reduced normal equations did not appear explicitly in any treatise upon Least Squares prior to the third edition of W. Jordan's Vermessungskunde,* for reference to which I am indebted to Mr. Harold Jacoby. In the first volume of this work, which bears the separate title "Ausgleichungs-Rechnung," Jordan, at page 64, calls the equations of the form

$$r = ax + by + cz + \ldots + lt - n \tag{5}$$

"Fehlergleichungen," and the results of eliminating x by means of the normal equation for x, in the form

$$r = b'y + c'z + \ldots + l't - n', \tag{6}$$

"Reduzierte Fehlergleichungen." He proceeds to show, as above, that the reduced normal equations are the normal equations which would arise from these reduced error equations, or the corresponding observation equations (3).† Since we can proceed in the same manner to the second reduced error equations and normal equations, and so on, it is obvious that we thus prove directly that

* The first edition appeared under the title "Taschenbuch der praktischen Geometrie," Stuttgart, 1874. The second edition was called "Handbuch der Vermessungskunde," Stuttgart, 1877. The third edition, improved and enlarged, appeared in 1888.

† The idea appears in a less explicit form in Herr und Tinter's "Lehrbuch der Sphärischen Astronomie," 1887, p. 47.

$[vv] = [nn, \mu]$, without the intervention of the equation

$$[vv] = [nn] - [an]x - [bn]y - \ldots - [ln]t\,.$$

It would seem, however, that this aspect of the reduced normal equations must have been present to the mind of Gauss while writing Art. 182 of the "Theoria Motus," in which he discusses the function W which is $[vv]$ considered as a quadratic function of $x, y, z, \ldots, t$, or Σv^2 when the v's are in the form (5).* If $X = 0$ is the normal equation for x, it is shown that $W_1 = W - \dfrac{X^2}{[aa]}$ is a function independent of x; and, since when $X = 0$ it has the same value as W, it is the result of eliminating x from W by means of the first normal equation. It is then shown that, $Y_1 = 0$, $Z_1 = 0$, etc. being the reduced normal equations, Y_1, Z_1, etc. have the same relation to W_1 that X, Y, etc. have to W. In like manner $W_2 = W_1 - \dfrac{Y_1^2}{[bb, 1]}$ is independent of y, and so on. Supposing there are four unknown quantities, the function W is thus finally reduced to the form

$$W = \frac{X^2}{[aa]} + \frac{Y_1^2}{[bb, 1]} + \frac{Z_2^2}{[cc, 2]} + \frac{T_3^2}{[dd, 3]} + \text{constant.}\dagger$$

In the "Disquisitio de Elementis ellipticis Palladis" a somewhat different investigation of this function is given in which the characteristic Gaussian notation is used for the first time, and the constant is shown to be $[nn, \mu]$. Gauss here for brevity suppresses the demonstration that $[bb, 1]$, $[cc, 2]$, etc. as well as $[aa]$ are positive, which is necessary to the proof that $[nn, \mu]$ is the minimum value of $[vv]$. But in the "Theoria Motus" he proves, for another purpose, that $[bb, 1]$ is a sum of squares, "for," he says, "it is evident that W_1 is derived from $[vv]$ the quantity x being eliminated from $v_1, v_2, \ldots$ by means of the equation $X = 0$; hence $[bb, 1]$ will be the sum of the coefficients

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* Slight changes have been made in the notation.

† This result is then used in finding the probability of a value of t. In the original edition of the "Theoria Motus" it is said that the probability that s (the fourth variable) shall differ from its most probable value by the quantity σ is proportioned to $e^{-\frac{hh\sigma\sigma}{\delta'''}}$ (δ''' here standing for $[dd, 3]$). This was a mistake for $e^{-hh\delta'''\sigma\sigma}$, and the same mistake is repeated two lines below where $\sqrt{\frac{1}{\delta'''}}$ instead of $\sqrt{\delta'''}$ is stated to be the measure of the relative precision to be attributed to the determination. These mistakes are corrected in Bertrand's tranlation, entitled "Methode des Moindres Carrés," p. 129, except that by a misprint the δ has dropped out of the exponential leaving the meaningless expression $e^{-hh'''\sigma^2}$. By a remarkable coincidence we find also in Davis's translation of the "Theoria Motus" $e^{-hh'''\sigma\sigma}$. Indeed there seems to have been a strange fatality connected with this passage; for in the "Gesammelte Werke" the exponential is printed $e^{-hh\delta'''s}$, introducing two new misprints.

of y^2 in $r_1^2, r_2^2, \ldots$ after the elimination, each of which is a square." Again Encke, in his elaboration of Gauss's method (*Berliner Jahrbuch* for 1835, p. 277) remarks, for the same purpose, that "if in each r the value is given to x for which $X = 0$, the sum of the squares of these r's is $W - \dfrac{X^2}{[aa]}$, and therefore the coefficient of y^2, which is $[bb, 1]$, will be simply a sum of squares."

In other words, W_1 is the value of Σr^2 when each r is expressed in the form (6), so that the coefficient of y^2 in it is $\Sigma b'^2$ or $[b'b']$, which is therefore equivalent to $[bb, 1]$.

Among the advantages of this point of view in treating the reduced normal equations, we may mention not only the ready proof that $[bb, 1]$, $[cc, 2]$, etc. are positive, and that $[nn, \mu]$ is the value of $[vv]$ when $v_1, v_2, \ldots$ are the residuals, but the simplification it introduces into Gauss's elegant discussion of the general value of the sum of the squares of the errors, and also its bearing upon the weights of the unknown quantities.

It is noteworthy that if, after solving the μ normal equations obtained from m observation equations, we should discover that the μ unknown quantities were connected by a rigorous equation of condition which had previously been overlooked, the most probable values of the unknown quantities would be changed *even if the values we had already found for the unknown quantities happened to exactly satisfy the rigorous equation of condition.* At first sight the rigorous relation being exactly satisfied would seem to be a confirmation of the values already obtained; but this is evidently not true, the normal equations used in obtaining them have now no standing, the true process being to eliminate one of the unknown quantities from the observation equations by means of the rigorous condition, and then form $\mu - 1$ normal equations for the remaining unknown quantities. We thus have a system of μ equations for the μ quantities only one of which is by hypothesis satisfied by the values already obtained.

But suppose the rigorous equations of condition happened to coincide with one of the normal equations first found, say that for x. Then, in accordance with what has been shown above, the new normal equation, being formed from observation equations identical with the fictitious observation equations described above (x being the unknown quantity first eliminated) are the same as the reduced normal equations. They therefore lead to the same values of $y, z, \ldots, t$ as before, and hence also to the same value of x.

Moreover we are led also to the same values of the weights of $y, z, \ldots, t$.*

* The value of $[vv]$ remains also unchanged, but the mean error of an observation is decreased, since there are now but $\mu - 1$ unknown quantities and we have to divide $[vv]$ by $m - \mu + 1$ instead of $m - \mu$.

The weight of x is however increased. For supposing, as in the usual process for obtaining the weights, the values of $x, y, \ldots, t$ when fully expanded in terms of the n's to be

$$x = \alpha_1 n_1 + \alpha_2 n_2 + \ldots + \alpha_m n_m\,,$$
$$y = \beta_1 n_1 + \beta_2 n_2 + \ldots + \beta_m n_m\,,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

the inverse weight of x is $\Sigma\alpha^2$. The normal equation for x may be written

$$x = \frac{[an]}{[aa]} - \frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \ldots;$$

whence equating the coefficients of n_1,

$$\alpha_1 = \frac{a_1}{[aa]} - \frac{[ab]}{[aa]}\beta_1 - \frac{[ac]}{[aa]}\gamma_1 - \ldots.$$

Now if the value of x above is derived from a rigorous equation, the first term is independent of the observed values, and they only enter the value of x through the values of $y, z, \ldots$; whence the coefficient of n_1 becomes

$$\alpha'_1 = -\frac{[ab]}{[aa]}\beta_1 - \frac{[ac]}{[aa]}\gamma_1 - \ldots.$$

and the inverse weight becomes $\Sigma\alpha'^2$. We have, then,

$$\alpha_1 = \frac{a_1}{[aa]} + \alpha'_1\,;$$

whence

$$\Sigma\alpha^2 = \frac{\Sigma a^2}{[aa]^2} + 2\,\frac{\Sigma a\alpha'}{[aa]} + \Sigma\alpha'^2$$
$$= \frac{1}{[aa]} + \Sigma\alpha'^2\,,$$

since $\Sigma a\alpha'$ consists of terms involving $\Sigma a\beta$, $\Sigma a\gamma$, etc. each of which is known to vanish. Thus the inverse weight is less and the weight greater than before.

SOME PROPOSITIONS CONCERNING THE GEOMETRIC REPRESENTATION OF IMAGINARIES.

By Dr. Maxime Bôcher, Cambridge, Mass.

The present paper contains a few propositions concerning the geometric representation of the roots of equations. The principal proposition contained in Sec. 1 is due to Lucas who gave a mechanical proof for it. I do not know whether the geometrical proof here given has been published before. I have found no reference to the propositions of Sec. 2.

1. Let $f(z) = 0$ represent an algebraic equation of the nth degree in z, and $f'(z) = 0$ its first derivative with regard to z. We shall make no assumption concerning the reality of the coefficients of $f(z) = 0$, but shall not consider the case in which all the roots of this equation are real. We will denote these n roots by $z_1, z_2, \ldots, z_n$ and the $n - 1$ roots of the derived equation by $z'_1, z'_2, \ldots, z'_{n-1}$. All of these quantities will in general be complex, and we will represent them in the ordinary way by points in a plane. The object of the present paper is to discuss the geometric relation which the system of $n - 1$ points $z'_1, \ldots, z'_{n-1}$ bears to the system of n points $z_1, \ldots, z_n$.

We will first prove the following generalization of Rolle's theorem:

If a convex polygon be formed whose vertices are the outermost of the points $z_1, \ldots, z_n$, *this polygon will include all the points of the system* $z'_1, \ldots, z'_{n-1}$.*

To prove this we will prolong any one of the sides AB of the polygon indefinitely in both directions, and take that direction upon it as positive in which it must be described in order that the rest of the polygon should lie to its left. We will prove that no point z which lies to the right of this line can be a root of the equation $f'(z) = 0$. Let us denote by θ the angle measured from the positive direction of the axis of reals to the positive direction of AB, and by φ the argument of the complex quantity $f(z)$. Writing, now,

$$f'(z) = \frac{f(z)}{z - z_1} + \frac{f(z)}{z - z_2} + \ldots + \frac{f(z)}{z - z_n},$$

it is easy to see that the argument of each of the terms in the second member lies between $\varphi - \theta$ and $\varphi - \theta - 180°$. These terms will therefore be represented by points lying to the right of the line through the origin, which makes an angle of $\varphi - \theta$ with the axis of reals. Their sum can therefore not be zero,

* Exceptions will occur not merely when all the points $z_1, \ldots, z_n$ lie on a straight line, this being practically the case of real roots; but also when $f(z) = 0$ has multiple roots, for then some of the points $z'_1, \ldots, z'_{n-1}$ will lie not *within* the polygon but at its vertices.

and z cannot be a root of $f'(z) = 0$. All the points $z'_1, \ldots, z'_{n-1}$ must then lie on the same side of the line AB as the polygon itself. The line AB was however any side of the polygon, whence it follows that the points $z'_1, \ldots, z'_{n-1}$ all lie within the polygon.

The theorem just proved is however merely a qualitative one, and we naturally wish to determine quantitatively the relation between the two systems of points.

Writing the equation $f(z) = 0$ in the form

$$z^n + nA_1 z^{n-1} + \frac{n(n-1)}{2!} A_2 z^{n-2} + \ldots + n A_{n-1} z + A_n = 0,$$

we have the relation

$$\frac{z_1 + z_2 + \ldots + z_n}{n} = -A_1 = \frac{z'_1 + z'_2 + \ldots + z'_{n-1}}{n-1},$$

from which we can infer at once the simple proposition,

The centre of gravity of the points $z_1, \ldots, z_n$, coincides with the centre of gravity of the points $z'_1, \ldots, z'_{n-1}$.

In Sec. 2 I have attempted to supplement this obvious proposition by others of a similar nature.

2. We will first consider the special case in which $f(z) = 0$ is the general cubic equation

$$z^3 + 3A_1 z^2 + 3A_2 z + A_3 = 0;$$

and ask ourselves, How are the two points z'_1, z'_2 related to the triangle whose vertices are the points z_1, z_2, z_3? A natural supposition is that the two points z'_1, z'_2 are the foci of some conic simply related to this triangle. The centre of such a conic must, according to the closing proposition of Sec. 1, lie at the centre of gravity of the triangle. Now one conic whose centre lies at the centre of gravity of a triangle is the ellipse tangent to the sides of the triangle at their middle points, or, as it is often called from one of its properties, the maximum ellipse inscribed in the triangle. We are thus naturally led up to the following proposition :—

The points z'_1, z'_2 are the foci of the maximum ellipse inscribed in the triangle whose vertices are z_1, z_2, z_3.

In order to prove this theorem I will first show that the points z'_1, z'_2 possess a certain property of the foci in question. This property depends upon the fact (see Salmon's Conic Sections, p. 182) that a tangent to an ellipse from a point P makes the same angle with the line joining P to one of the foci as the line joining P to the other focus makes with the other tangent from P. Here we will take as the point P any of the vertices of the triangle, for instance

z_3. F_1 and F_2 being the foci in question, our proposition tells us that the angles $z_1 z_3 F_1$ and $F_2 z_3 z_2$ are equal. Now, we can prove directly that the angles $z_1 z_3 z'_1$ and $z'_2 z_3 z_2$ are equal, for they are the arguments of the complex quantities $\frac{z'_1 - z_3}{z_1 - z_3}$ and $\frac{z_2 - z_3}{z'_2 - z_3}$. In order to prove them equal we have, then, merely to show that the quotient of these two quantities is real and positive, and this quotient is

$$\frac{(z'_1 - z_3)(z'_2 - z_3)}{(z_1 - z_3)(z_2 - z_3)} = \frac{z'_1 z'_2 - z_3(z'_1 + z'_2) + z_3^2}{z_1 z_2 - z_3(z_1 + z_2) + z_3^2} = \frac{A_2 + 2A_1 z_3 + z_3^2}{3A_2 + 6A_1 z_3 + 3z_3^2} = \frac{1}{3}.$$

Now with the points z'_1, z'_2 as foci describe an ellipse tangent to one side of the triangle. Then, by the closing proposition of Sec. 1 its centre will coincide with the centre of gravity of the triangle, and by the proposition just proved the ellipse will also be tangent to the other two sides of the triangle. Therefore, since only one conic can be drawn with a given point as centre and tangent to three given lines, this ellipse must be the maximum ellipse and the points z'_1, z'_2 must be the foci of the maximum ellipse.*

A part of the proposition thus proved may without difficulty be extended as follows to the case where $f(z) = 0$ is an equation of the nth degree :—

If z_i is any of the points $z_1, \ldots, z_n$, and if the rest of these points are paired in any way with the points $z'_1, \ldots, z'_{n-1}$, then the sum of the angles subtended by these pairs of points at the point z_i is equal to zero (the angles being measured in each case from the unaccented to the accented member of the pair).

To which we may add :

The product of the distances from z_i to the other points $z_1, \ldots, z_n$ is n times the product of the distances from z_i to the points $z'_1, \ldots, z'_{n-1}$.

Could not the first of these propositions be brought into connection with the focal properties of the higher plane curves (see Salmon's Higher Plane Curves § 142)?

Finally we may remark that the function $f'(z)$ being merely the first polar of the function $f(z)$ with regard to the point $z = \infty$ we can pass from the case above considered to the general case where we have to deal with the roots of the first polar of $f(z)$ with regard to any point P by means of a fractional linear transformation. Several of the above propositions can easily be enunciated so as to suit this more general case by replacing straight lines by circles through P, conics by bicircular quartics with double point at P, etc.

* We obtain incidentally the following geometrical proposition :—

The product of the lines joining the foci of the maximum ellipse with any of the vertices of the triangle is one-third the product of the sides of the triangle adjacent to this vertex.

NOVEMBER 18, 1892.

ON THE CONE OF SECOND ORDER WHICH IS ANALOGOUS TO THE NINE-POINT CONIC.

By Mr. Thomas F. Holgate, Worcester, Mass.

The theorem on the existence of the so-called nine-point conic may be stated in its most general form as follows:—

Let $ABCD$ be any complete quadrangle whose six sides, AB, AC, AD, BC, BD, CD are cut by an arbitrary straight line a in the points P, Q, R, S, T, V, respectively; and let E, F, H, K, L, M be the harmonic conjugates of these points with respect to the pairs of vertices of the quadrangle, so that $AEBP$, $AFCQ$, etc., are harmonic ranges. Then a conic may be passed through the six points E, F, H, K, L, M, on which will also lie the three points of intersection, X, Y, Z, of the pairs of opposite sides of the quadrangle.

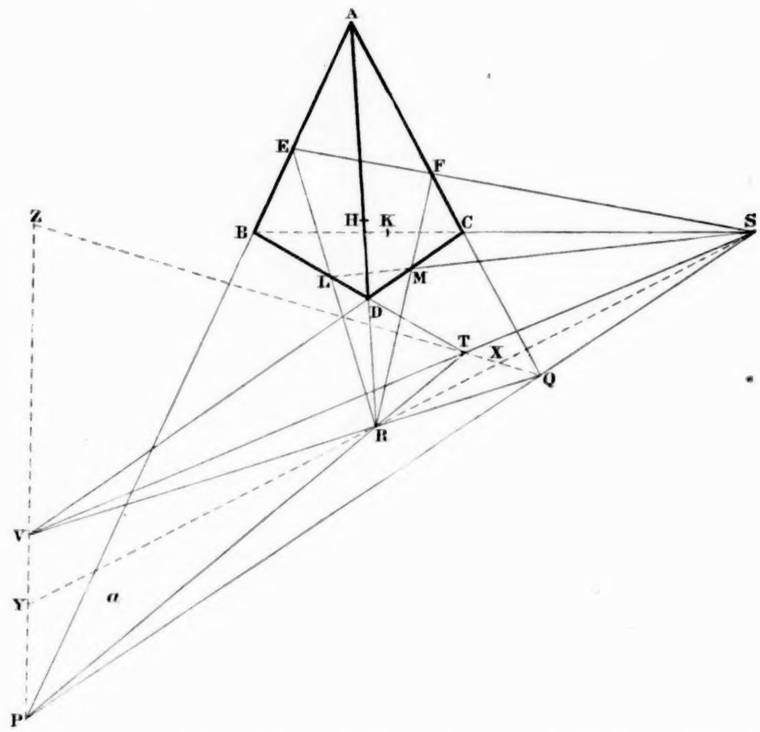
In this form the theorem admits of a very simple geometrical demonstration, and embraces as special cases the ordinary nine-point conic and nine-point circle. For, if the line a be the infinitely distant line in the plane, the harmonic conjugate points become the bisection points of the sides of the quadrangle, and if at the same time the quadrangle be such that the three pairs of opposite sides intersect at right angles, the conic becomes a circle.

Since the line a may have any position in the plane, it determines the doubly infinite system of conics through the three points X, Y, Z. If it pass through one of these points, for example through X, the corresponding conic degenerates into two straight lines of which the one passes through X, and the other is the line YZ. If the line a pass through two of these points, say X and Y, then the corresponding conic breaks up into the two straight lines XZ and YZ.

We readily obtain an analogous relation in space of three dimensions by a study of the following configuration:—

Suppose $ABCD$ be any tetrahedron whose faces are cut by an arbitrary plane a, not passing through any of the vertices, and without loss of generality we may suppose a to cut all the faces extended. Let a cut the edges AB, AC, AD, BC, BD, CD, in P, Q, R, S, T, V, respectively, and to each of these points find the harmonic conjugate with respect to the pair of tetrahedral points lying on the same edge. Let these conjugate points be E, F, H, K, L, M, so that $AEBP$, $AFCQ$, etc. are harmonic ranges along the edges of the tetrahedron. Then the following theorem holds true:—

From any point O in the plane a, these six points, E, F, H, K, L, M, may be projected by a cone of the second order, which however breaks up into two



planes for certain special positions of the vertex; and, if from O be drawn the three rays meeting pairs of opposite edges of the tetrahedron, these will also lie on this cone.

1. The points P, Q, R, S, T, V lie by threes in straight lines, the intersections of the faces of the tetrahedron with a. Moreover, S, F, and E are collinear. For, since $S(AEBP)$ is a harmonic pencil lying in the plane ABC, and is cut by the straight line $AFCQ$ so that C and Q lie respectively in the rays SB and SP, then must F fall in the ray SE since by supposition $AFCQ$ is a harmonic range. So also S, M, and L are collinear, as are all sets of similarly situated points.

Consider E, F, H, M, L, K to be a gauche hexagon with EF and ML, FH and LK, HM and KE, as pairs of opposite sides. Let O be any point in space, and from it project the hexagon.

The planes OEF and OML will intersect in the line OS,
" OFH " OLK " " OV,
" OHM " OKE " " OQ.

By the space extension of Pascal's Theorem, a cone of the second order whose vertex is O will circumscribe the hexagon when and only when OS, OV, and

OQ lie in one plane. These lines are coplanar if O lies in the plane SVQ, i. e. in the plane a.

2. The points group themselves by fours in planes, as follows:

$EFLM$ in a plane cutting	a	in the line	RS,	
$EHKM$ " "	a	" "	TQ,	
$FHLK$ " "	a	" "	PV,	

Hence, out of any point in one of the lines RS, TQ, or PV, the six points are projected by two planes, one of which contains four of the points, the other, the remaining two points. It is also easily seen that the six points are projected by two planes, three points in each plane, out of any point in the four lines of intersection of the faces of the tetrahedron with a, so that when the vertex is a point in any one of these seven lines, which are the four sides and three diagonals of a complete quadrilateral, the cone of projection breaks up into two planes.

3. Let OG be the ray drawn from the vertex of the cone to meet any pair of opposite edges of the tetrahedron, for example, AB in G_1 and CD in G_2.

Then the planes OG_1E and OFK intersect in the line OP,

OEH " OKM " " OT,

OHF " OMG_2 " " OV.

But since O lies in the plane PTV, or a, the lines OP, OT, OV are coplanar, and hence the ray OG, and likewise each of the other two rays similarly drawn, lies on the cone of second order determined by the five rays $O(E, H, F, K, M)$. But this is the same cone on which we have shown the ray OL to lie. Hence the theorem.

Since E, F, L, M lie in one plane, EM and FL, lines joining the points in opposite edges of the tetrahedron, must intersect. Similarly, EM and HK, HK and FL intersect. As these lines EM, HK, and FL are not coplanar and each intersects the other two, they must pass through one point. As an interesting special case let a be parallel to one of the faces of the tetrahedron, for example, BCD. Then K, L, M become the middle points of the edges in this face, while E, F, H lie in a plane parallel to this face. Therefore,

If the three edges of a tetrahedron which lie in one face be bisected and the remaining three edges be cut by a plane parallel to this face, then the three lines joining the points so marked out in the pairs of opposite edges intersect in one point, and conversely.

If the plane a move out to infinity we have the well known theorem that "the joined lines of the middle points of the three pairs of opposite edges in any tetrahedron meet in a point." In this case also we observe that *the six lines which are drawn parallel to one another, in any direction, through the middle points of the edges of any tetrahedron and the three lines parallel to these*

which meet the pairs of opposite edges of this tetrahedron, all lie on a cylinder of the second order.

In the general case, the point of intersection, W, of the three joined lines of the points marked out in the pairs of opposite edges bears many interesting relations to the plane α with reference to the tetrahedron. To each plane α in space is correlated in this way but one point W, and conversely. In such a system of correlation, however, the faces and edges of the tetrahedron to which reference is made, are singular planes and lines, since to any face all the points in that face are correlated, and to a point in any edge, all the planes through that edge are correlated.

From what has already been shown it readily follows, that

If there be given any six points in space which lie in pairs on three straight lines through a point W, a plane α may be determined out of any point of which the six points are projected by a cone of the second order, viz., the plane through the harmonic conjugates of W with respect to the three pairs of points.

The plane α is correlated to the point W not only with reference to the tetrahedron $ABCD$, but also with reference to what may be called the anti-tetrahedron; viz., that whose faces are determined by the points EHF, FKM, MLH, EKL. The faces and edges of this tetrahedron cut α in the same lines and points as do those of the original tetrahedron. Moreover, the choice of this plane α, to which the point W is correlated, determines, at the same time, three other planes β, γ, δ, and three other points X, Y, Z, which bear the same relation to each other, respectively, as do α and W. The three points X, Y, Z lie in the plane α, and are those points in which the lines EM, FL, HK meet the plane. Thus we obtain a new tetrahedron $WXYZ$ which is correlated to $ABCD$ in such a manner that if from any one of its vertices, for example, from Y, lines be drawn to meet the three pairs of opposite edges of $ABCD$, the six points so marked out on these edges are projected from any point in the opposite face, WXZ, by a cone of second order, on which will also lie the rays drawn from this point of projection to meet the pairs of opposite edges of the original tetrahedron.

CLARK UNIVERSITY, *May*, 1892.

THE CONJUGATION-POINT AND THE APPLICATION OF ITS PROPERTIES TO GRAPHIC CONSTRUCTIONS.

By Prof. Frank H. Loud, Colorado Springs, Colo.

The theorem which I propose to state, and to apply to some simple problems of the geometry of the conic, is itself readily deduced from principles relating to involutions, well known to every student of the projective geometry. But in this form it is new, so far as I am aware; and as I think its practical applications may render it serviceable to teachers of elementary analytical geometry, I shall state and prove it in terms adapted to classes in that branch.

Definition.—If through the centre of a given conic there be drawn (1) the circumference of a circle, (2) a tangent to that circumference, which tangent will of course be a diameter of the conic, (3) a second diameter, conjugate to the one just mentioned,—then the point on this second diameter where it is intersected by a line joining those two points on the circumference of the circle in which the latter meets the two axes of the conic separately is called the *conjugation-point* of the conic in respect to the circle.

To determine the coordinates of the conjugation-point.—The definition suggests the convenient method of finding these, which would doubtless be taken unaided by the elementary student. If the conic be given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the coordinates of the centre of the circle be m, n, then the circle and lines named in the definition have successively the equations, $x^2 - 2mx + y^2 - 2ny = 0$, $mx + ny = 0$, $b^2nx - a^2my = 0$, and $nx + my - 2mn = 0$. By elimination from the last two the required coordinates are found $x = \frac{2a^2m}{a^2 + b^2}$, $y = \frac{2b^2n}{a^2 + b^2}$. To adapt the formula to the hyperbola requires of course a change of the sign of b^2.

Theorem.—The conjugation-point of a conic with respect to a circle is in the line joining the two points in which the circle is cut by any pair of conjugate diameters of the conic separately.

To prove this theorem, using the formula just obtained, we have first to assume an equation to represent any diameter, which, for the sake of symmetry, may be taken to be $bsx - ay = 0$ then that of the conjugate will be $bx + asy = 0$. Then the two points which we have to show to be in one line with the point $\frac{2a^2m}{a^2 + b^2}, \frac{2b^2n}{a^2 + b^2}$ are found to be $\frac{2a^2m + 2abns}{a^2 + b^2s^2}, \frac{2abms + 2b^2ns^2}{a^2 + b^2s^2}$ and $\frac{2a^2ms^2 - 2abns}{a^2s^2 + b^2}, \frac{-2abms + 2b^2n}{a^2s^2 + b^2}$. If, now, this must be shown by ele-

mentary algebraic methods, using the formula of the common text-books, the process is indeed somewhat tedious, but of course in no other sense difficult. But if determinants may be used, the proof is short enough. For, in this case, the formula

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

readily takes the form

$$\frac{4ab}{(a^2+b^2)(a^2+b^2s^2)(a^2s^2+b^2)}\begin{vmatrix} am & bn & a^2+b \\ am+bns & ams+bns^2 & a^2+b^2s^2 \\ ams^2-bns & -ams+bn & a^2s^2+b^2 \end{vmatrix}.$$

Now, if for the second or third row be substituted the sum of these two, divided through by $1+s^2$, the determinant is found to have two rows identical.

Problem.—Having given in position, (not in length), two diameters of a conic and the conjugate of each, to construct, in position, the conjugate of any other given diameter, the axes of the curve, and (if the latter be an hyperbola) the asymptotes. (The conic itself is supposed not to be drawn.)

Draw, through the intersection of the given diameters, the circumference of any convenient circle, producing, if necessary, all the diameters to meet the latter. Join the point in which any diameter meets the circumference with that in which its conjugate meets the same; two lines are thus found, on each of which lies the conjugation-point; the latter is therefore at their intersection. Join the point in which the fifth given diameter meets the circle with the conjugation-point, and the point in which this line meets the circle again with the centre of the conic; this line is the required conjugate diameter. (It is on account of the facility with which a conjugate is found to a given diameter that I have suggested the name *conjugation-point.*) To locate the axes, join the centre of the circle to the conjugation-point; the points in which this line cuts the circumference are on the axes respectively. For the lines from these points to the centre are conjugate diameters as above; they also make an angle which is right, because inscribed in a semicircle. If the conic whose diameters are given is an ellipse, the conjugation-point will be found within the circle of construction; if it be an hyperbola, without. From this point draw tangents to the circle, and connect the points of contact with the centre of the conic; these lines are the asymptotes. For each is a diameter conjugate to itself.

The foregoing is sufficient to show the convenience with which this point may be used in constructions.

COLORADO COLLEGE, *June* 6, 1892.

MACFARLANE'S ALGEBRA OF PHYSICS.*

By Prof. E. W. Hyde, Cincinnati, Ohio.

Before entering upon a careful consideration of Professor Macfarlane's paper, I wish to express my hearty appreciation of the skill and ability shown by him in the development of a method of multiplication so interesting from an algebraic point of view, as well as in his analysis of the fundamental principles of the subject. The weighty objections to the system of Quaternions as developed by Hamilton are well and strongly stated, and the Author seeks to develop a system which shall possess what he regards as the advantages of Hamilton's without its drawbacks. The recognition and separate treatment of products and quotients of vectors as things essentially different is certainly a great improvement over the quaternion treatment.

The opening sentences of the paper appear to indicate that the Author has been partly moved by a desire to vindicate the work of Hamilton and Tait against those who have asserted the superiority of Grassmann's system.

The Author claims to have devised "*a more complete algebra which unifies Quaternions, Grassmann's Method, and Determinants, and applies to physical quantities in space.*" It is proposed to examine this claim somewhat in detail. As to the last specification, there is no doubt of the *fact*; the only question is as to relative superiority of the different systems.

The defect asserted with regard to Grassmann's system is that in it we do not have a *complete* distributive product, because the product of identical factors is zero.† In ordinary algebra we have $(a+b+c)(a'+b'+c') = aa' + ab' + ac' + ba' + bb' + bc' + ca' + cb' + cc'$, and we should have a similar result in directional algebra.

But do we not have that very thing? Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be any six vectors; then by the *Ausdehnungslehre* $(\alpha+\beta+\gamma)(\alpha'+\beta'+\gamma') = \alpha\alpha' + \alpha\beta' + \alpha\gamma' + \beta\alpha' + \beta\beta' + \beta\gamma' + \gamma\alpha' + \gamma\beta' + \gamma\gamma'$, precisely as in scalar algebra, and, in general, none of the partial products is zero. Now suppose the quantities $a, \ldots, c'$, to vary continuously according to definite laws; then they will, in general, pass one or more at a time, through the value zero. At the instant of such passage of any one of the quantities three of the partial products disappear. Does that vitiate the multiplication? Similarly let $\alpha, \ldots, \gamma'$ vary. Now, 1°, their

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* Principles of the Algebra of Physics. By A. Macfarlane, M. A., D. Sc., LL. D. A paper read before the A. A. A. S. at the Washington Meeting, and published in Vol. XL of the Proceedings.

† See bottom of p. 77.

tensors can change while their directions remain constant, giving precisely the same results as with $a, \ldots, c'$, and, 2°, their *directions* can change while the tensors are constant. In this case among the infinite number of sets of simultaneous values a certain comparatively small number will cause one or more partial products to vanish because of parallelism. Why should this form a valid objection to this kind of multiplication? 3°. Both lengths and directions of the vectors may change, which simply combines the two previous results. It appears then that we *do* have in Grassmann's system a perfectly general and complete distributive product.

At the bottom of p. 77 it is asserted that "as a consequence of not treating together the two complementary parts of the product of two vectors, Grassmann and his followers have *restricted their attention to associative products*" (Italics mine). The fact is that Grassmann's multiplication is only *partially* associative: what he calls progressive and regressive products are such, but mixed products are *not*. Thus, if p be a point, and L_1 and L_2 be point-vectors (fixed in position), then $pL_1 . L_2$ is not the same as $p . L_1L_2$, in either two or three dimensional space. In the same place it is said "that is a very arbitrary principle which holds that all the terms into which two similar directions enter must vanish." It is difficult to see in this anything *more* arbitrary than in the assumption that the product of a vector by itself should be the square of its tensor. Grassmann's result is obtained by the most natural and reasonable extension of the idea of the square of a scalar, and represents perfectly the fact that if a cell with four equal sides be closed up by making two opposite angles zero and the other two each 180°, the enclosed area vanishes.

Towards the bottom of p. 71 occurs the following remark: "This kind of product, in which the factors are vectors, has in recent times been generally neglected." A statement is then quoted from Clifford to the effect that in any product all factors but the last are *operators*. Now the combinatory products of Grassmann are essentially of the kind spoken of by the Author as neglected, $\alpha\beta$ being the product of the two independent vectors α and β, and not at all the operation α on β, and so for all products.

Let us now consider the claim of the Author to have devised "a more complete algebra which unifies Quaternions, Grassmann's method, and Determinants." We have already seen that Grassmann's products have the complete distributive quality as much as Professor Macfarlane's; we will now look at the two systems from a geometric standpoint. To be sure, our Author styles his work the "Algebra of Physics;" but what do we treat in Physics by means of our equations but the geometric relations of the quantities involved? Now, there enter into Grassmann's system *all* the geometric quantities that exist, as the point, fixed line, fixed plane, line-direction, and plane-direction

for three-dimensional space, and similarly for space of four, five, or n dimensions; and we have a complete system of geometric multiplication involving all of these, which adapts the method wonderfully to the purposes of geometry. In the proposed system we have only the line direction as in Quaternions, which is to be supplemented by the notation $A . F$ for a fixed line, A and F being simply directed lines, or vectors. Now, of course, we can if we choose regard this as meaning that A is drawn out from some fixed point, and then F through the end of it; but is it quite logical to combine two things, of which neither is fixed in position, and call the combination a *fixed* vector? In the *Ausdehnungslehre*, on the contrary, p is a definite point whose multiplication into the definite direction ε make the definite line $p\varepsilon$ through p in the direction ε; and this line we may, and do constantly, represent by a single letter as L, using either notation as may be convenient. The expression $A . F$ is thus used to designate the same thing as $p\varepsilon$, but can p therefore be properly said to be "equivalent" to A, as stated on p. 80? It is equivalent only in *use*, not in *meaning;* in the one case one quantity having only length and direction is arbitrarily used to fix in *position* another quantity having the same qualities, while in the other the idea of position in p is combined with those of direction and length in ε. It is difficult to see how such a notation could be used with any facility in treating of the screws, and wrenches, and twists of Professor Ball; while Grassmann's system works to perfection in this field.

Now, in view of these facts, can the proposed system be regarded as more complete than Grassmann's, even with reference only to three-dimensional space? What is done, as regards the product of two vectors, is simply to take as the, so-called, *complete* product the sum of Grassmann's *outer* and *inner* products, a complex expression, since one of these quantities is scalar and the other directed. The complexity is not practically helped by regarding the scalar part as having an indefinite axis, as is done by Professor Macfarlane, though it may be a relief to one's intellectual scruples. In the case of three or more vectors the results are far more complex, and seem to me to be of little utility. I am obliged to confess that the Author's arguments fail to convince me of the necessity or desirability of using this "complete" product. He says on p. 76, "The works of Hamilton and Tait make it abundantly evident that the quaternion idea is essential to the algebraic treatment of Spherical Trigonometry and of rotation." It is certainly true that the use of the "quaternion idea," i. e. the versor, or ratio of two vectors, simply as a versor, or turner, is convenient in treating rotations; but this can be done equally well with Grassmann' system as with Hamilton's or with Professor Macfarlane's, and does not at all involve the necessity of making the product of two or more vectors complex. Further, all the fundamental formulæ of Spherical Trigo-

nometry are derived with the utmost ease by Grassmann's methods without using the versor, as well as those of Plane Trigonometry except De Moivre's; though there is no particular practical advantage in so doing, because students rarely, if ever, study directional algebra till they have mastered Trigonometry.

Another word with reference to point analysis. Professor Macfarlane uses the equation $p_2 = p_1 + \varepsilon$ to show that Grassmann's system involves heterogeneous equations, and argues that, if ε is a vector, p_1 and p_2 must be vectors also, so that the point-analysis becomes really a vector-analysis. (See p. 79.) The fact is, the case is just the other way, and ε is a *point*, viz. a point of zero weight at ∞, and is *always* to be so regarded when it appears in an equation with points.

Now what do we know about such a zero point at ∞? Simply its *direction*, and the fact that it changes the position of the mean point of any system of points to which it may be added by a certain definite distance in its own direction, on account of its infinite *arm*. This distance in a definite direction is the essence of a vector, which assigns the reason for frequently calling p_∞ a vector.

We have a precisely similar thing in forces and couples. A couple is simply the sum of two forces equal in magnitude, parallel, and opposite; it must thus be of the same kind, and is, in fact, a zero force at ∞, and yet it appears as of two dimensions when regarded simply as a couple.

Arguing in the same line (p. 79) the Author says, "From the physical point of view it is more correct to treat of a mass-vector than of a point having weight; for the differential coefficient with respect to time of a mass-vector is the momentum, which is itself a mass-vector. If the latter is of one dimension in length so is the former. The product of a point and a mass is not a physical idea." But the differential coefficient with respect to time of a weighted point is *likewise* the momentum, the differential of a point being always a vector, and the argument falls to the ground. As to the product of a point and a mass not being a physical idea; we certainly deal in physics with masses situated at points, and why should not the combination of the two be regarded as a product as much as the combination of a length and a direction?

Towards the bottom of p. 87 the Author makes the following statement: "By the complementary vector (Fig. 12) of A with respect to B, Grassmann means the vector which has the same magnitude as A and is drawn perpendicular to A in the plane of A and B." This is certainly a mistake. There is no such thing as a line-vector complementary to a line-vector in Grassmann's system in space of higher dimensions than *two*. In three-dimensional space, which is here under consideration, the complement of a vector is a perpendic-

ular *plane*-vector whose tensor (area) is equal numerically to the tensor (length) of the vector, which is a very different conception. The vector is of *one* dimension, its complement is of *two*. This brings out again the fact that *only* line-vectors enter into Hamilton's or Macfarlane's systems, whereas in four-dimensional space we should have solid-vectors, as well as plane-vectors, and so on for spaces of higher orders. To be consistent with his own notation Professor Macfarlane should write $\overline{ij} = k$ instead of $ij = k$, etc., thus avoiding an apparent heterogeneity.

In view of what has been adduced it appears to me that Professor Macfarlane's claims for his method can not be regarded as valid. He has shown conclusively, in my judgment, that the combination of the vector and the versor in one and the same symbol, as was done by Hamilton, is neither necessary nor desirable, and has worked out a consistent system without that combination, treating separately product and ratio (quaternion) quantities, which may be regarded as an improvement on Hamilton's method ; but, in the opinion of the writer of this article, the methods of the *Ausdehnungslehre* so far excel those of Quaternions in completeness, generality, simplicity, and ease of interpretation and application, especially in geometry, but also in the realm of physics, that no idea of *unifying* the two can possibly be entertained.

HAGEN'S SYNOPSIS OF HIGHER MATHEMATICS.*

By Prof. William Benjamin Smith, Columbia, Mo.

One very grave obstacle to the successful prosecution in the United States of original research in mathematics has been the practical inaccessibility of its literature. Ignorance of this literature, which is immense, has meant ignorance of the results of previous and contemporaneous investigation the world over, as well as of the new outstanding problems disclosed from time to time, no less than of the directions in which energy might be hopefully expended. But while the vast bulk of this literature lay strewn without order in two-score sets of periodicals, and for the most part in foreign languages, it remained a hopeless task for the great body of our youth bent on mathematical study even to use it with much advantage, and far more hopeless to master it. Excellent service to the cause of learning has of late years been rendered in England, and still more upon the Continent, in collecting the widely-scattered memoirs of the great masters and publishing complete editions of their works. This indeed is much, but it is yet far from being enough. A book is worth nothing that is not worth an index, and if in the whole circuit of intellectual achievement there be any results that deserve careful collation and orderly arrangement, it must be in the Higher Mathematics. Even the sight of but one wing or transept in the temple of mathematical knowledge, as of Modern Geometry or the Invariantive Analysis, is awe-inspiring, but how much more a view, as from some higher space, of the entire edifice, not merely as one huge mass but in all the details of its finely articulated structure. Now it is precisely such a view as this, minute and yet comprehensive, that Prof. Hagen has attempted to present in his monumental *Synopsis der Höheren Mathematik*. In four stately volumes he seeks to summarize the results of as many centuries of investigation, and enable the student and explorer to ascertain, without tedious consultations of dispersed and often unobtainable memoirs, the state of knowledge at any given point, to orient himself anywhere, to demark sharply the known from the unknown, and to note the trend and promise of the lines of advancing discovery. A more useful labor than this in the present condition of mathematical literature can hardly be imagined; moreover, it calls for all but the very highest, that is, creative mathematical power; in particular, for

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* Synopsis der Höheren Mathematik. Von Johann G. Hagen, S. J., Director der Sternwarte des Georgetown College, Washington, D. C. Erster Band: Arithmetische und Algebraische Analyse. Berlin, Verlag von Felix L. Dames.

immense erudition, an unerring logical instinct for the often extremely subtile relations obtaining among propositions; but above all, for untiring industry. Once accomplished, however, such a work would be of permanent value, and would lay all future generations, no less than the present, under heavy obligations. It was with lively pleasure, then, that we read the prospectus of Prof. Hagen's undertaking, and the first volume, now before us, meets fully our anticipations. It is an imposing quarto of four hundred pages bearing the imprint of Felix L. Dames, Berlin, and is concerned exclusively with *Arithmetische und Algebraische Analyse*. It falls into twelve *Abschnitte*, each devoted to a *Theorie*, as of *Numbers*, *Complex Magnitudes*, *Combinations*, *Series*, *Product-series and Faculties*, *Continued Fractions*, *Sums and Differences*, *Functions*, *Determinants*, *Invariants*, *Substitution-groups*, and lastly, *Equations*. The first, *Theorie der Zahlen*, is embraced in forty-two pages. When we reflect that Gauss regarded the Theory of Numbers as the very core of Mathematics, in fact almost as Mathematics itself, and all else bearing the name as auxiliary thereto, or at best an application thereof, we are at first surprised to find only one-tenth of one volume out of four dedicated to this section. However, the second section, of twelve pages, carries us into the modern doctrines of Complex Numbers, Gaussian, Kummerian, ideal, and universal, and so in a measure restores a juster proportion. Moreover, not a little that one person might range under the Theory of Numbers another might assign to the Theory of Equations or to Combinatorial Analysis; and we do, in fact, find many properties of numbers in these two sections. Only a very minute inspection could show, then, that this leading division of the work had been slighted. Besides all this, the profounder theories of Kummer and Kronecker, of Dedekind and Weierstrass, have been only broached as yet, and are very far from complete development.

Again, we find forty-two pages devoted to the *Theory of Functions*. At once we think of the Theta-functions of Jacobi, and of the Weierstrassian *Sigma* and *Pe* functions, creations that of themselves have widely extended the range of mathematical power, and we wonder whether the author has succeeded in tabulating in forty-two pages such a vast body of doctrine. But the title-page reminds us that this volume treats only of Arithmetical and Algebraic Analysis, and accordingly only so much of *Function-Theory* is here summarized as belongs properly to Algebra. For the great creations of the last two generations in Elliptic and Hyperelliptic Functions we must wait on the fourth volume. Hence it appears that any criticism upon any apparent deficiency or inadequacy must run great risk of turning out to be premature. The principles of division adopted in the text may or may not recommend themselves to the reader; but in such matters there is ample room for diversity of judgment.

Nowhere else in the whole range of human intellectual endeavor are doctrines the most various so interpenetrative as in Mathematics. Each discipline is indeed, so to say, a personality, existing in and for itself, yet each is inextricably interlaced with every other. The realm of Mathematics is thus a conform depiction of the universe itself, a veritable chart of Nostradamus,

> Wo Alles sich zum Ganzen webt,
> Eins in dem Andern wirkt und lebt.

It is not strange, then, that several classifications may have equal logical justification, and what we do not find where we first seek it may with good reason be introduced elsewhere. We have not read all of these four hundred quarto pages, but we have consulted them repeatedly on almost every variety of subject they contain, and though occasionally disappointed at first, never without reasonable satisfaction in the end. The work abounds in cross references that are very convenient, while the references to original authorities are exceedingly helpful. *Scire ubi aliquid inveniendum, magna pars scientiæ.* The historical data interspersed throughout the book appear to have great value. One always feels much nearer to a theorem, knowing its date, its discoverer, and some of its antecedents. We also thank the author for his *Vorbemerkungen*, which are in general very judicious.

Examining the work we seem to be looking at some precious Florentine mosaic, where the *disjecta membra* of four centuries have been so patiently collected and skillfully composed as to present the appearance of a single organic whole. In reality the work has been the slow and toilsome deposition of a quarter-century. It is impossible, then, that all parts should show equal exhaustiveness, if indeed equal carefulness. Some of the more ancient results are accessible in a hundred well-known volumes; some of the more modern perhaps in only a single rare number of an obscure periodical. It is not surprising, then, if the *Theory of Invariants* appear less fully treated than such a subject as the *Theory of Equations*. But on the whole, the completeness of the exhibit is admirable; and the immense agglomerate of doctrine has been fused into a well-nigh homogeneous mass. Throwing open the book at various select critical points we find a number of pencil marks made on first reading, where the statements, if not incorrect, seem at least inadequate, or unguarded, or perhaps pronounced from a standpoint already overcome. But inaccuracies lie concealed in the most conscientious writings, and it would be a fine sieve that should strain out every one in transference to such a *Synopsis*. Some of these have already caught the author's eye and are noted among the *Berichtigungen*, along with a few errors of type. Of these latter also, however, some remain uncorrected; as on page 7, line 2, where a and b should be α and β. But the general mechanical execution is excellent.

This is preëminently an age of great mathematicians. Not even the younger sciences, as Biology, are advancing more rapidly than their eldest sister. If the radius of the sphere of mathematical lore be not augmenting faster than ever, at least the volume certainly is. Three more tomes, companions of this under review, are promised. Before even diligence can hurry them through the press, great additions will surely have been made to our present treasures, and a large supplement will be needed. Elegantly dedicating his high emprise *Almæ Georgiopolitanæ Academiæ*, the author piously invokes Divine aid to her noble work of culture, as she auspiciously opens her second *sæculum;* nor perhaps would a similar sentiment unfittingly close this inadequate notice. Assuredly every lover of learning must pray that Prof. Hagen may carry forward his work to completion even beyond the wide confines of its original conception. But however this may be, what has already been accomplished is of great and lasting value, and establishes a secure claim to the gratitude of all students of Mathematics.

UNIVERSITY OF MISSOURI, *September* 19, 1892.

SOLUTIONS OF EXERCISES.

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330

FIND the sum of the series

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \dots + [\tfrac{1}{2}n(n+1)]^2.$$

[*Artemas Martin.*]

SOLUTION.

Putting S for the sum sought we have by the method of differences,

$$S = n + \frac{n(n-1)}{1.2}d_1 + \frac{n(n-1)(n-2)}{1.2.3}d_2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}d_4 + \dots,$$

where d_1, d_2, d_3, d_4, etc. are the first terms of the first, second, third, fourth, etc. orders of differences.

The given series, expanding the terms, is

$$1, \ 9, \ 36, \ 100, \ 225, \ 441, \ 784, \ 1296, \ \dots,$$

and the differences are

$$\begin{array}{cccccccc}
8, & 27, & 64, & 125, & 216, & 343, & 512, & \dots, \\
 & 19, & 37, & 61, & 91, & 127, & 169, & \dots, \\
 & & 18, & 24, & 30, & 36, & 42, & \dots, \\
 & & & 6, & 6, & 6, & 6, & \dots, \\
 & & & & 0, & 0, & 0, & \dots.
\end{array}$$

Hence $d_1 = 8$, $d_2 = 19$, $d_3 = 18$, $d_4 = 6$, $d_5 = 0$. Substituting these values in the expression for S and reducing, we find

$$S = \tfrac{1}{60}n(n+1)(n+2)(3n^2 + 6n + 1).$$

[*Charles Yardley.*]

331

The extremities of a diameter of a variable ellipse having fixed foci lie on a fixed hyperbola having the same foci; show that the extremities of the conjugate diameter lie on another hyperbola having the same foci.

[*W. Woolsey Johnson.*]

SOLUTION.

Let the equation of the fixed hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

An ellipse having the same foci may be represented by the equation

$$\frac{x^2}{a^2 + \lambda^2} + \frac{y^2}{\lambda^2 - b^2} = 1 .$$

For their points of intersection,

$$x^2 = \frac{(a^2 + \lambda^2)a^2}{a^2 + b^2}, \quad y^2 = \frac{b^2(\lambda^2 - b^2)}{a^2 + b^2} .$$

If x', y' be the coordinates of the extremity of the conjugate diameter,

$$x'^2 = y^2 \cdot \frac{a^2 + \lambda^2}{\lambda^2 - b^2}, \quad y'^2 = x^2 \cdot \frac{\lambda^2 - b^2}{a^2 + \lambda^2};$$

whence $\frac{x'^2}{b^2} - \frac{y'^2}{a^2} = 1$, a confocal hyperbola.

[*Geo. R. Dean.*]

337

Required the locus of the foot of the perpendicular from the centre of an ellipse upon the common chord of the ellipse and circle of curvature.

[*Artemas Martin.*]

SOLUTION.

The equation to the chord of curvature, as given in Salmon's Conics, p. 229, Ex. 4, is

$$\frac{x}{a} \cos a - \frac{y}{b} \sin a = \cos 2a . \tag{1}$$

The perpendicular from the centre on this is

$$x = -\frac{b}{a} y \cot a ;$$

whence

$$\tan a = -\frac{by}{ax} .$$

Substituting in (1) the values of $\cos a$, $\sin a$, $\cos 2a$ obtained from this, we have

$$\frac{x^2}{\sqrt{a^2x^2+b^2y^2}}+\frac{y^2}{\sqrt{a^2x^2+b^2y^2}}=\frac{a^2x^2-b^2y^2}{a^2x^2+b^2y^2},$$

or clearing of fractions and radicals,

$$(x^2+y^2)^2(a^2x^2+b^2y^2)=(a^2x^2-b^2y^2)^2.$$

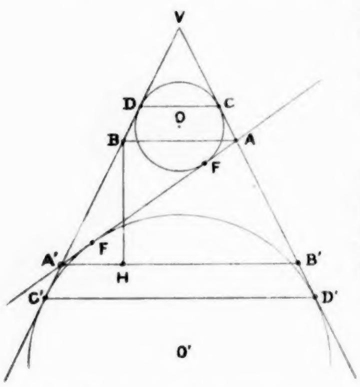
[*W. B. Richards.*]

338

PROVE synthetically that the eccentricity of a conic section is equal to the sine of the angle which the cutting plane makes with the base of the cone, divided by the sine of the angle which an element of the cone makes with the base. [*H. B. Newson.*]

SOLUTION.

Let $VC'D'$ be the axial section of the cone and AA' the cutting plane. Let O and O' be spheres inscribed in the cone touching the plane AA' at the points F and F', and let CD and $C'D'$ be the circles where the spheres O and O' touch the cone.



Pass the planes AB and $A'B'$ through A and A' parallel to the planes of the circles CD and $C'D'$. Draw the line BH perpendicular to $A'B'$. Let the angle $AA'B$ be a, and $BA'B'$ be θ.

It can be easily proved that F and F' are the foci of the conic, and that the line DC' is equal to the major axis AA' (see *Salmon's Conic Sections*, Art. 367); whence

$$DC'=AA'. \tag{1}$$

But $$AC=AF, \tag{2}$$

and $$A'C'=A'F', \tag{3}$$

since they are tangents to a sphere from the same point.

Subtracting (2) and (3) from (1), we obtain

$$DA'-CA=FF';$$

or, since $$CA=DB,$$

$$A'B=FF'.$$

Now $$e = \frac{FF'}{AA'} = \frac{BA'}{AA'},$$

and $$BA' = \frac{BH}{\sin\theta}, \quad AA' = \frac{BH}{\sin\alpha};$$

whence $$e = \frac{\sin\alpha}{\sin\theta}.$$ Q. E. D.

[*H. C. Riggs.*]

343

Show that

$$\sum_{m=0}^{m=n} (-1)^m \frac{2^{2m-1}-1}{(2n-2m+1)!} \frac{B_m}{(2m)!} = 0,$$

where B_m represents Bernoulli's number.

[*W. H. Echols.*]

SOLUTION.

We know that

$$\frac{x}{e^x+1} = \frac{x}{2} - (2^2-1)\frac{B_1}{2!}x^2 + (2^4-1)\frac{B_2}{4!}x^4 - (2^6-1)\frac{B_3}{6!}x^6 + \dots,$$

and

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \frac{B_1}{2!}x^2 - \frac{B_2}{4!}x^4 + \frac{B_3}{6!}x^6 - \dots.$$

Adding these, we obtain

$$\frac{xe^x}{e^{2x}-1} = \frac{1}{2} - (2^1-1)\frac{B_1}{2!}x^2 + (2^3-1)\frac{B_2}{4!}x^4 - (2^5-1)\frac{B_3}{6!}x^6 + \dots.$$

But we know that

$$\frac{e^{2x}-1}{2xe^x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \dots.$$

Multiplying these two last series together, since they are absolutely convergent, we obtain

$$0 = \left[-\frac{2^1-1}{2!}B_1 + \frac{1}{2}\frac{1}{3!}\right]x^2 + \left[\frac{2^3-1}{4!}B_2 - \frac{2^1-1}{3!}\frac{B_1}{2!} + \frac{1}{2}\frac{1}{5!}\right]x^4 + \dots$$

$$+ x^{2n}\sum_{m=0}^{m=n}(-1)^m \frac{2^{2m-1}-1}{(2n-2m+1)!}\frac{B_m}{(2m)!} + \dots,$$

which establishes the formula. [*W. H. Echols.*]

EXERCISES.

352

A solid sphere and a solid cylinder of equal radii roll from rest down the same inclined plane; compare the times of their descent.

[*Artemas Martin.*]

353

Prove that, in Weierstrass's notation,

$$\sigma u\, \sigma_3 v\, \sigma_2 w + \sigma_3 u\, \sigma v\, \sigma_1 w + \sigma_2 u\, \sigma_1 v\, \sigma w = 0\,,$$

where $u + v + w = 0$.

[*Frank Morley.*]

354

Lagrange's interpolation formula is

$$fx = \sum^{n} A_n\, fa_r + (-1)^n (x - a_1) \ldots (x - a_n) \frac{f^n(u)}{n!}\,,$$

wherein u is some value of x which lies between the greatest and least of the values x, $a_1, \ldots, a_n$, and $f^n(u)$ means that fx is to be differentiated n times and in the result u put in the place of x; and

$$A_r = \Pi \frac{(x - a_i)}{(a_r - a_i)}\,. \qquad (i = 1, \ldots, r - 1, r + 1, \ldots, n)$$

If $a_r = a^r x$, we have

$$fx + \sum_1^n A_r\, fa^r x = (a - 1) \ldots (a^n - 1) \frac{x^n}{n!} f^n(u)\,,$$

$$A_r = (-1)^r \frac{a^r(1 - a^{-(n-r+1)}) \ldots (1 - a^{-n})}{(a - 1) \ldots (a^r - 1)}\,.$$

Interpret this result when a lies between $+1$ and -1, and also when $a^2 > 1$ and $n = \infty$.

[*W. H. Echols.*]

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Required the locus of the point in the normal to a conic, which is equally distant from the focus and the foot of the normal.

[*Geo. R. Dean.*]

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